Second-order linear differential equations with constant coefficients

bikobi

1 ***Homogeneous*** 2nd-order linear DEs with constant coefficients

You want to solve a second-order DE of the following form ($a \neq 0$):

$$ay'' + by' + cy = 0 \tag{1}$$

Try $y = e^{rt}$ as a solution, just because that's what the solution to the *first*-order DE looks like. So find y' and y'':

$$\begin{aligned} y' &= re^{rt} \\ y'' &= [y']' = r^2 e^{rt} \end{aligned}$$

and plug them into Equation 1:

$$a \cdot r^2 e^{rt} + b \cdot re^{rt} + c \cdot e^{rt} = 0$$

Now factor out the e^{rt}:

$$e^{rt} \cdot (ar^2 + br + c) = 0$$

You know that e^{rt} is never zero for any value of t, so the expression above is true only if:

$$ar^2 + br + c = 0$$

The values of r_1, r_2 can be found with the quadratic formula:

$$r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{D}}{2a}$$

The determinant $D = b^2 - 4ac$ can be either positive, negative, or zero.

1.1 Case I: $D > 0$

In this case, r_1 and r_2 are real and distinct. This means that $y_1 = e^{r_1 t}$ and $y_2 = e^{r_2 t}$ are two particular solutions to Equation 1, and the *general* solution is given by their linear combination:

$$\boxed{y = A \cdot e^{r_1 t} + B \cdot e^{r_2 t}} \tag{2}$$

1.2 Case II: $D = 0$

In this case, $r_1 = r_2$, so $y = e^{rt}$ is a solution to Equation 1.

To find the *general* solution, let $y = e^{rt} \cdot u(t)$ (with $u(t)$ a generic function to be determined) and try that as a solution. (This method is called a "reduction of order"; it's used find a second, *linearly independent* solution from a solution that is already known, in this case $y_1 = e^{rt}$).

$$\begin{aligned} y' &= r \cdot e^{rt} \cdot u + e^{rt} \cdot u' = e^{rt} \cdot (ru + u') \\ y'' &= e^{rt} \cdot r \cdot (ru + u') + e^{rt} \cdot (ru' + u'') \\ &= e^{rt} \cdot (r^2 u + ru' + ru' + u'') \\ &= e^{rt} \cdot (r^2 u + 2ru' + u'') \end{aligned}$$

Plug into Equation 1:

$$
\begin{aligned}
a \cdot e^{rt} \cdot (r^2u + 2ru' + u'') + b \cdot e^{rt} \cdot (ru + u') + c \cdot e^{rt} \cdot u = 0 \\
ae^{rt}r^2u + ae^{rt}2ru' + ae^{rt}u'' + be^{rt}ru + be^{rt}u' + ce^{rt}u = 0 \\
e^{rt} \cdot (ar^2u + a2ru' + au'' + bru + bu' + cu) = 0 \\
e^{rt} \cdot (au'' + (2ar + b)u' + (ar^2 + br + c)u) = 0
\end{aligned}
$$

Now, you know that:

- $e^{rt} \neq 0$ for all t.
- $ar^2 + br + c = 0$ because r is root of the auxillary equation.
- $2ar + b = 0$ because, when $D = 0$, then $r = \frac{-b}{2a} \implies 2ar + b = 0$

So:

$$
\begin{aligned}
e^{rt} \cdot (au'' + \cancelto{0}{(2ar + b)u'} + \cancelto{0}{(ar^2 + br + c)u}) = 0 \\
e^{rt} \cdot au'' = 0 \xrightarrow[\substack{e^{rt} \neq 0 \\ a \neq 0}]{} u'' = 0
\end{aligned}
$$

Thus the equation is reduced to $u'' = 0$ ($a \neq 0$ otherwise Equation 1 wouldn't be second-order). From this, compute the integral twice and find that $u = A + Bt$ for arbitrary constants A and B.
The general solution was defined as $y = e^{rt} \cdot u(t)$, therefore:

$$
y = e^{rt} \cdot (A + Bt)
$$

$$
\boxed{y = Ae^{rt} + Bte^{rt}} \tag{3}
$$

1.3 Case III: $D < 0$

The two roots are given by

$$
r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{D}}{2a}
$$

When $D < 0$, that means:

$$
\begin{aligned}
D = -|D| \\
\sqrt{D} = \sqrt{-|D|} = i \cdot \sqrt{|D|}
\end{aligned}
$$

which, when plugged in the quadratic formula, gives:

$$
r_1, r_2 = \frac{-b \pm i\sqrt{|D|}}{2a} = k \pm i\omega \qquad \text{where } k = \frac{-b}{2a}, \quad \omega = \frac{\sqrt{4ac - b^2}}{2a}.
$$

Therefore, there are two *complex* solutions to Equation 1:

$$
\begin{aligned}
y_1^* = e^{k+i\omega} \\
y_2^* = e^{k-i\omega}
\end{aligned}
$$

In order to obtain two *real* solutions, conveniently combine y_1^* and y_2^* and use Euler's relation:

$$
e^{ix} = \cos x + i \sin x
$$

$$
\begin{aligned}
y_1 &= \frac{1}{2}y_1^* + \frac{1}{2}y_2^* = \frac{1}{2}(e^{(k+i\omega)t} + e^{(k-i\omega)t}) = \frac{1}{2}(e^{kt}e^{i\omega t} + e^{kt}e^{-i\omega t}) = \frac{1}{2}e^{kt}(e^{i\omega t} + e^{-i\omega t}) \\
&= \frac{1}{2}e^{kt}[(\cos\omega t + i\sin\omega t + \cos\omega t - i\sin\omega t] = \frac{1}{2}e^{kt}2\cos\omega t = \boxed{e^{kt}\cos\omega t}
\end{aligned}
$$

$$
\begin{aligned}
y_2 &= \frac{1}{2i}y_1^* - \frac{1}{2i}y_2^* = \frac{1}{2i}(e^{kt}e^{i\omega t} - e^{kt}e^{-i\omega t}) = \frac{1}{2i}e^{kt}(e^{i\omega t} - e^{-i\omega t}) \\
&= \frac{1}{2i}e^{kt}(\cos\omega t + i\sin\omega t - \cos\omega t + i\sin\omega t) = \frac{e^{kt}}{2i}2i\sin\omega t = \boxed{e^{kt}\sin\omega t}
\end{aligned}
$$

So y_1, y_2 are two real solutions to Equation 1. Their linear combination is then the *general* solution:

$$
\boxed{y = A \cdot e^{kt}\cos\omega t + B \cdot e^{kt}\sin\omega t} \tag{4}
$$

2 *Non-homogeneous* 2nd-order linear DEs with constant coefficients

You want to solve a second-order DE of the following form ($a_2 \neq 0$; $f(x) \neq 0$):

$$
a_2(x)y'' + a_1(x)y' + a_0(x)y = f(x) \tag{5}
$$

Assume that you know two independent solutions to its associated (or complementary) *homogeneous* equation, $y_1(x)$ and $y_2(x)$ (the section above explains how to find them). Then their linear combination $y_h = C_1y_1 + C_2y_2$ is the *general* solution to the complementary homogeneous equation, and it is called the *complementary function* of the non-homogeneous DE Equation 5.

The general solution to the non-homogeneous DE Equation 5 is

$$
y = y_p(x) + y_h(x)
$$

where $y_p(x)$ is *any* particular solution to the non-homogeneous DE (due to the principle of superposition for linear ODEs). Therefore, to solve Equation 5, all you need is any particular solution. There are two methods to find one.

2.1 Method of undetermined coefficients

This methods consist of making an educated guess for the form of the solution you're looking for, based on the non-homogeneous term. This generic form contains some unknown coefficients; the generic form of the candidate solution with unknown coefficients is then plugged into the non-homogeneous DE, and the resulting equation is solved for the unknown coefficients.

Sometimes, it is necessary to add an "extra degree of freedom" to the candidate solution, because without it it becomes impossible to solve for the coefficients. This happens when the candidate solution is also a solution to the complementary homogeneous equation.

There are some appropriate forms to try as candidate particular solutions, in Table 1 below:

f	candidate y_p
d (constant)	α (constant)
$ex + d$	$\alpha x + \beta$
$e^{\mu x}$	$\alpha e^{\mu x}$
$\cos(\omega x)$ or $\sin(\omega x)$	$\alpha\cos(\omega x) + \beta\sin(\omega x)$

Table 1: If any term of y_p is a solution of the homogeneous equation, multiply y_p by x (or by x^2 if necessary).

2.2 Method of variation of parameters

Another, more formal method to find a particular solution to a second-order linear non-homogeneous DE with constant coefficients, when you already know two independent solutions to the complementary homogeneous equation, is to replace the *constants* in the complementary function by *functions*. That is, search for a y_p in the form

$$
y_p = u_1(x)y_1(x) + u_2(x)y_2(x) \tag{6}
$$

Requiring y_p to satisfy the non-homogeneous DE provides one equation that must be satisfied by the two functions u_1 and u_2. You can require them to satisfy a second, arbitrary equation, to have enough pieces of information to solve for u_1 and u_2 (a system of two equations with two unknowns). To simplify future calculations, this second arbitrary equation is chosen to be

$$u_1'y_1 + u_2'y_2 = 0$$

Now you have:

$$\begin{aligned} y_p &= u_1y_1 + u_2y_2 \\ y_p' &= u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2' \xrightarrow[\text{arbitrary equation}]{} y_p' = u_1y_1' + u_2y_2' \\ y_p'' &= u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2'' \end{aligned}$$

which you can plug into Equation 5:

$$\begin{aligned} a_2y'' + a_1y' + a_0y &= f(x) \\ a_2(u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2'') + a_1(u_1y_1' + u_2y_2') + a_0(u_1y_1 + u_2y_2) &= f(x) \\ a_2u_1'y_1' + a_2u_1y_1'' + a_2u_2'y_2' + a_2u_2y_2'' + a_1u_1y_1' + a_1u_2y_2' + a_0u_1y_1 + a_0u_2y_2 &= f(x) \\ a_2(u_1'y_1' + u_2'y_2') + u_1(a_2y_1'' + a_1y_1' + a_0y) + u_2(a_2y_2'' + a_1y_2' + a_0y_2) &= f(x) \\ a_2(u_1'y_1' + u_2'y_2') &= f(x) \end{aligned}$$

Thus, the two unknown functions u_1 and u_2 satisfy the two equations

$$\begin{cases} u_1'y_1 + u_2'y_2 = 0 \\ u_1'y_1' + u_2'y_2' = \frac{f(x)}{a_2} \end{cases}$$

The system can be solved (e.g. with Cramer's rule):

$$u_1' = -\frac{y_2(x)}{W(x)}\frac{f(x)}{a_2(x)}$$

$$u_2' = \frac{y_1(x)}{W(x)}\frac{f(x)}{a_2(x)}$$

where $W(x)$ (the *Wronskian* of y_1 and y_2) is the determinant:

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}$$

At this point, u_1 and u_2 are found by integration and plug into Equation 6.